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ANALYSIS OF WAVE MOTIONS IN ONE CRYSTALLOGRAPHIC PLANE OF A CUBICALLY ANISOTROPIC MEDIUM

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Based on the system of the equations of motion of an elastic medium with tetragonal anisotropy, a comparative analysis of the curves of inverse velocities and the elastic-wave fronts in one crystallographic plane of a cubically anisotropic body has been made with the use of sections of the surfaces of inverse velocities and wave surfaces.

Investigations of surfaces characterizing different aspects of propagation of elastic waves occupy an important place in the dynamic theory of elasticity of anisotropic media. Such surfaces include, for example, the surfaces of inverse velocities, radiation surfaces, etc. [1, 2]. This is due to the fact that they allow not only calculation of the velocity of propagation of waves and the rate of transfer of energy but also clear representation of wave processes and description of their features. In view of the complexity and cumbersomeness of the corresponding characteristic and dispersion equations, however, such investigations are frequently confined to construction of the sections of the surfaces of inverse velocities and radiation surfaces in special planes of anisotropic media that are for the most part coordinate in the basic crystallographic coordinate system of an anisotropic body. Data on the regularities of propagation of elastic waves in planes that are not coordinate have been given, for example, in [2]. In particular, the dependences of the velocities of propagation of quasilongitudinal and quasitransverse waves in a cubically anisotropic medium in one plane of the auxiliary coordinate system ($\bar{x}_1$, $\bar{x}_2$, x_3) have been analyzed based on the system of equations of motion of tetragonally anisotropic media (Fig. 1). Below, we propose a comparative analysis of these results and the results of investigation of the curves of inverse velocities and the wave fronts in the plane $\bar{x}_1 = 0$ of the coordinate system ($\bar{x}_1$, $\bar{x}_2$, x_3) of a cubically anisotropic medium in the context of three-dimensional representations of wave motions [3].

Following [4], we write the expressions for dimensionless velocities of propagation of elastic waves in the plane $\bar{x}_1 = 0$ of a cubically anisotropic medium, which follow from the corresponding system of the equations of motion [2]:

$$\bar{v}_{1,2} = \sqrt{\frac{1}{2}\left(1 + \left(1 + \frac{a+b}{2}\right)\cos^2\alpha + (a+1)\sin^2\alpha \pm \Omega\right)}\ ,$$

$$\Omega = \sqrt{\left(\frac{a+b}{2}\cos^2\alpha - a\sin^2\alpha\right)^2 + b^2\sin^2 2\alpha}\ , \tag{1}$$

$$\bar{v}_3 = \sqrt{\frac{a-b+2}{2}\sin^2\alpha + \cos^2\alpha}\ .$$

In formula (1) and in what follows, the subscript 1 corresponds to the quasilongitudinal wave and the subscripts 2 and 3 correspond to the quasitransverse waves. The absolute value of the velocity is determined by multiplication of v_i by c.

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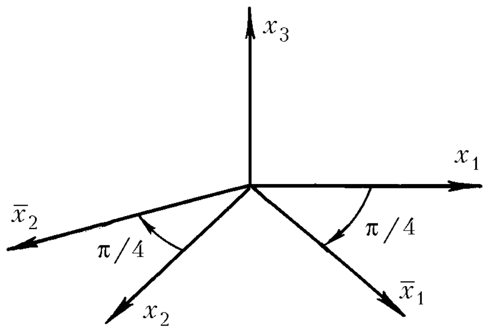

Fig. 1. Basic (x_1, x_2, x_3) and auxiliary ($\bar{x}_1$, $\bar{x}_2$, x_3) coordinate systems.

The coordinates of points of the plane $\bar{x}_1 = 0$ of the cubically anisotropic medium that have been reached by a wave disturbance by the time $t = 1$ sec will be obtained with the use of bicharacteristics [4] in dimensionless form:

$$\bar{x}_2^{(1,2)} = \frac{\sin\alpha}{v_{1,2}}\left(2+a\pm\frac{1}{\Omega}\left(2b^2\cos^2\alpha-4a\left(\frac{a+b}{2}\cos^2\alpha-a\sin^2\alpha\right)\right)\right),$$

$$x_3^{(1,2)} = \frac{\cos\alpha}{2v_{1,2}}\left(4+a+b\pm\frac{1}{\Omega}\left(4b^2\cos^2\alpha+(a+b)\left(\frac{a+b}{2}\cos^2\alpha-a\sin^2\alpha\right)\right)\right), \tag{2}$$

$$\bar{x}_2^{(1,2)} = \frac{(a-b+2)\sin\alpha}{2v_3}, \quad x_3^{(1,2)} = \frac{\cos\alpha}{v_3}.$$

The absolute values of the coordinates of points of the wave front at the time t are found by multiplication of the dimensionless values given below in the figures by ct.

Using (1) and (2), we construct the dimensionless curves of inverse velocities $\bar{r}_2 = 1/\bar{v}_2$ and $\bar{r}_3 = 1/\bar{v}_3$ and the wave fronts $\bar{l}_2$ and $\bar{l}_3$ in the plane $\bar{x}_1 = 0$ for a cubically anisotropic material characterized by the constants $a = 2.24$ and $b = 3.72$ (the elasticity constants A_1, A_2, and A_4 have been taken from [5]).

From Fig. 2, it follows that the curve of inverse velocities $\bar{r}_3$ and the wave front $\bar{l}_3$ are ellipses; formulas (1) and (2) yield that the $\bar{r}_3$ and $\bar{l}_3$ curves have the form of an ellipse in any cubically anisotropic medium. Thus, from (2), after obvious transformations, we have

$$\frac{4\bar{x}_2^2}{(a+b-2)^2}+x_3^2=\frac{1}{\bar{v}_3^2}.$$

We consider the inverse-velocity curves and the wave fronts obtained by section of the corresponding surfaces by the plane $\bar{x}_1 = 0$. Omitting intermediate calculations, we give the expressions for the velocities of propagation of elastic waves and the coordinates of points of the wave front [3]:

$$v_k = \sqrt{1+\frac{a}{3}-2\sqrt{-\frac{p}{3}}\,\cos\left(\frac{\Lambda_k+2\pi k}{3}\right)}\,, \tag{3}$$

$$x_j^{(k)} = \frac{1}{\sqrt{1+\frac{a}{3}-2\sqrt{-\frac{p}{3}}\,\cos\left(\Lambda_k+2\pi k\right)}}\left(2\cos\alpha_j\left(1+\frac{a}{3}\right)+\right.$$

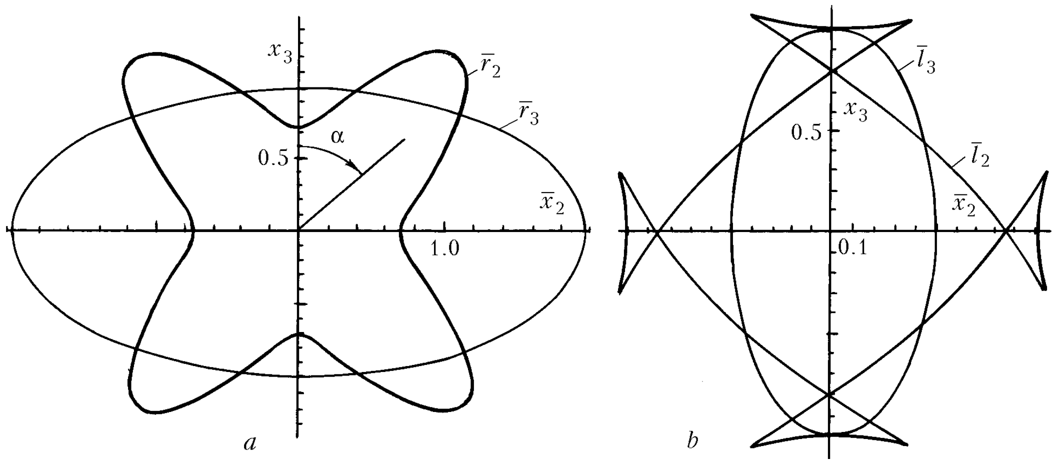

Fig. 2. Curves of inverse velocities $\bar{r}_2$ and $\bar{r}_3$ (a) and the wave fronts $\bar{l}_2$ and $\bar{l}_3$ (b) in the plane $\bar{x}_1 = 0$ of cubically anisotropic media according to [2].

$$+\sqrt{3}\left(\frac{p_j^*}{\sqrt{-p}}\cos(\Lambda_k + 2\pi k) + \frac{1}{3}\frac{\sqrt{4p^3}\sin(\Lambda_k + 2\pi k)}{\sqrt{4p^3 + 27q^2}}\sqrt{-\left(\frac{3}{p}\right)^3} \times \right. \tag{4}$$

$$\left. \times \left(q_j^* + \frac{9\sqrt{3}}{2}\frac{q}{p}p_j^*\right)\right)\Bigg),$$

where

$$\Lambda_k = \arccos\left(-\frac{q}{2}\sqrt{-\left(\frac{3}{p}\right)^3}\right); \quad p = (a^2 - b^2)\,m - \frac{a^2}{3};$$

$$q = \frac{2a^3}{27} - \frac{a\,(a^2 - b^2)\,m}{3} + (a^3 - 3ab^2 + 2b^3)\,n;$$

$$p_j^* = 2\,(a^2 - b^2)\cos\alpha_j\,(1 - \cos^2\alpha_j) - \frac{4a^2\cos\alpha_j}{3};$$

$$q_j^* = \frac{4a^3}{9} - \frac{2a}{3}(a^2 - b^2)\,(m\cos\alpha_j + \cos\alpha_j\,(1 - \cos^2\alpha_j)) +$$

$$+ 2\,(a^3 - 3ab^2 + 2b^3)\cos\alpha_j\,(m - \cos^2\alpha_j\,(1 - \cos^2\alpha_j));$$

$$m = \sum_{i \neq j = 1}^{3} \cos^2\alpha_i \cos^2\alpha_j; \quad n = \cos^2\alpha_1 \cos^2\alpha_2 \cos^2\alpha_3;$$

where the index k in formulas (3) and (4) points to the type of elastic wave.

Prescribing appropriately the direction cosines $\cos\alpha_j$ of the normal to the characteristic surface in (3) and (4), we can construct both three-dimensional surfaces and their sections by different planes. Figure 3 gives the curves of inverse velocities $\bar{r}_2$ and $\bar{r}_3$ and the wave fronts $\bar{l}_2$ and $\bar{l}_3$ in the plane $\bar{x}_1 = 0$ of the cubically anisotropic material whose elastic properties are characterized by the constants $a = 2.24$ and $b = 3.72$.

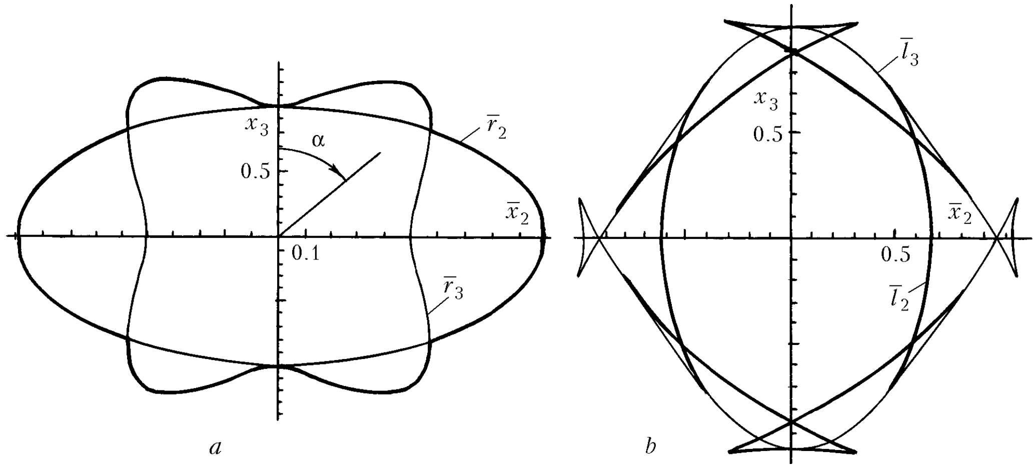

Fig. 3. Sections of the surfaces of inverse velocities (a) and the wave surfaces (b) of quasitransverse waves by the plane $\bar{x}_1 = 0$ for cubically anisotropic media.

From Fig. 3, it is clear that in the plane $\bar{x}_1 = 0$, the curves of inverse velocities $\bar{r}_2$ and $\bar{r}_3$ do not coincide with $\bar{r}_2$ and $\bar{r}_3$ constructed with the use of relations (1) (see Fig. 2). The wave fronts $\bar{l}_2$ and $\bar{l}_3$ obtained by section of the wave surfaces of quasitransverse waves also differ from $\bar{l}_2$ and $\bar{l}_3$ presented in Fig. 2. Thus, from Fig. 3 it follows that the propagation of the wave fronts $\bar{l}_2$ and $\bar{l}_3$ is accompanied by the formation of six and two lacunas respectively, whereas, according to Fig. 2 (in the plane $\bar{x}_1 = 0$ of the cubically anisotropic medium [2]), the wave front $\bar{l}_2$ has four lacunas and the front $\bar{l}_3$ is an ellipse and contains no lacunas. The analogous disagreements of the inverse-velocity curves and the wave fronts of quasitransverse waves, constructed based on the system of the equations of motion [2] and obtained by section of the corresponding surfaces by the $\bar{x}_1 = 0$ plane, can be found for cubically anisotropic materials characterized by other constants a and b. No differences have been established between the $\bar{r}_1$ and $\bar{l}_1$ curves constructed in the context of the above-described approaches for quasilongitudinal waves.

In closing, we consider certain features of the inverse-velocity curves; the presence of these features [1, 6] enables us to draw the conclusion on the occurrence of lacunas on the wave front. Among such features, according to [1], is the existence of two points having one tangent to the inverse-velocity curve. For example, to the $\bar{r}_2$ curve we can draw six tangents (see Fig. 3) having two such points on $\bar{r}_2$; therefore, the wave front of this quasitransverse wave must have six lacunas confined in the intervals of the angles between the rays through the origin of coordinates and the corresponding tangency points. This is confirmed by the front $\bar{l}_2$ given in Fig. 3.

Another approach is based on the general theory of curves of fourth order (the occurrence of lacunas on the wave front is suggested by the double points of inflection on the inverse-velocity curves) [6]. Thus, the $\bar{r}_2$ curves have two pairs of inflection points each, and the wave front $\bar{l}_2$ contains two lacunas lying in the same quadrants of the plane $(\bar{x}_2, x_3)$ as the portions of concavity on the inverse-velocity curves. However, as Fig. 3 shows, lacunas on the wave front $\bar{l}_2$ also occur in the absence of inflection points. Therefore, we can say that the presence of inflection points on the curve of inverse velocities is a necessary but insufficient condition for the occurrence of lacunas on the wave front.

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NOTATION

A_1, A_2, and A_4, elasticity constants of the cubically anisotropic medium; $a = A_1/A_4 - 1$; $b = A_2/A_4 + 1$; and $c = \sqrt{A_4/\rho}$; cos α_j, direction cosines of the normal to the wave surface; $\bar{r}_2$ and $\bar{r}_3$, inverse-velocity curves; t, time; α_j, slope of the wave normal to the x_i coordinate axis; ρ, density of the medium. Subscripts and superscripts: i and $k = \overline{1, 3}$.

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